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The temperature field in a cylindrical electric conductor with annular section

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Abstract—The heat conduction in a hollow cylindrical electric resistor which carries an alternating current is analysed. The hole within the cylinder is either empty or filled with a dielectric solid. The non-uniform power generated per unit volume in the resistor by the Joule effect is evaluated, and the steady periodic Fourier equation is written in a dimensionless form both in the domain occupied by the resistor and in that occupied by the dielectric, if present. A boundary condition of the third kind is assigned at the external surface of the cylinder. The dimensionless temperature field is determined analytically as a function of position, time and a proper set of dimensionless parameters.

INTRODUCTION

Heat transfer in electric resistors is a subject widely treated in the literature. In many textbooks on heat conduction (see for example Kakaç and Yener [1]), the evaluation of the temperature field in a solid cylinder crossed by a stationary electric current represents one of the simplest problems which involve the Fourier equation with a heat generation term. In that case, the solution can be easily determined because the temperature field is stationary and depends only on the radial coordinate, and because the heat generation term is uniform.

Nivoit *et al.* [2] have considered the dependence of the temperature field on the axial coordinate in the case of cylindrical electric wires with a very small section and crossed by a stationary current. These authors neglect the radial dependence of the temperature field because the section of the wire is small. They are concerned with the unicity of the stationary temperature distribution along the wire, especially in the case of radiation heat transfer through the external surface.

Other authors [3–6] have analysed the heat conduction in cylindrical electric resistors crossed by an alternating current. If the electric current is alternating, the current density has a non-uniform radial distribution within the resistor and, as the frequency increases, both the current density and the power generated per unit volume concentrate near the external surface. This circumstance is known in the literature as the *skin effect* and is discussed in textbooks on electrodynamics, for example in Landau and Lifshitz [7], for an infinitely long solid cylindrical resistor. Therefore, when the current is alternating the heat generation term to be included in the Fourier equation, which yields the temperature field within the resistor, is both non-uniform and time-dependent.

Thorn and Simpson [3] consider a hollow cylindrical resistor which is crossed by an alternating current and is surrounded both internally and externally by a vacuum, so that it radiates all the generated power. The time-averaged and dimensionless temperature difference between the internal and the external surface of the cylinder is evaluated for a fixed value of the outer radius and with a variable value of the inner radius.

Both in Owens [4] and in Morgan and Barton [5] a resistor with the shape of a solid cylinder and crossed by an alternating current is considered. Owens performs an analytical evaluation of the temperature field in the resistor by neglecting the skin effect, i.e. by assuming that the power generated per unit volume is uniform but time-dependent. On the other hand, Morgan and Barton take into account the skin effect and deal with the transient heat conduction in the resistor by employing a simplified Fourier equation in which the heat generation term is replaced by its time-average. These authors evaluate the time-evolution of the dimensionless surface temperature of the resistor immediately after the current has been switched on or off. Transient temperature distributions have also been studied by Sahin *et al.* [8], with reference to a slab of material, with an insulated boundary, that carries an alternating current during a direct resistance or an induction heating. These authors perform their analysis neglecting the time dependence of the power generated per unit volume by the Joule effect.

Recently, Barletta and Zanchini [6] have determined analytically the steady periodic temperature field in a solid cylindrical resistor, crossed by an alternating current with skin effect. The resistor is sup-

NOMENCLATURE

a	internal radius of the conductor [m]
A	integration constant introduced in equation (20)
b	external radius of the conductor [m]
B	integration constant introduced in equation (20)
Bi	Biot number hb/λ_c
c	speed of light in empty space [m s^{-1}]
$c_1, \ldots, c_4$	integration constants employed in equations (50) and (52)
C	constant defined by equation (90)
c_{p_c}	specific heat at constant pressure of the electric conductor [J kg^{-1} K^{-1}]
c_{p_d}	specific heat at constant pressure of the dielectric solid [J kg^{-1} K^{-1}]
$\mathbf{E}$	electric field [V m^{-1}]
E_0	modulus of $\mathbf{E}_0$ [V m^{-1}]
E_{0z}	component of $\mathbf{E}_0$ in direction z [V m^{-1}]
$\mathbf{E}_0$	space-dependent part of $\mathbf{E}$, defined by $\mathbf{E} = \mathbf{E}_0 e^{-i\omega t}$ [V m^{-1}]
f	dimensionless function of s, Ω and Δ defined in equation (32)
g	dimensionless function of s, Ω and Δ defined in equation (38)
h	convection coefficient [W m^{-2} K^{-1}]
$\mathbf{H}$	magnetic field [A m^{-1}]
$H_{0\theta}$	component of $\mathbf{H}_0$ in direction θ [A m^{-1}]
$\mathbf{H}_0$	space-dependent part of $\mathbf{H}$, defined by $\mathbf{H} = \mathbf{H}_0 e^{-i\omega t}$ [A m^{-1}]
i	imaginary unit $i = \sqrt{-1}$
I_{eff}	effective electric current (defined as $\lVert I \rVert/\sqrt{2}$, where I is the electric current) [A]
$\mathbf{J}$	electric current density [A m^{-2}]
J_n	Bessel function of first kind and order n
k_1, k_2	integration constants employed in equation (83)
Q	power generated per unit length by Joule effect [W m^{-1}]
q_g	power generated per unit volume by Joule effect [W m^{-3}]
$\overline{q_g}$	time-average of q_g [W m^{-3}]
r	cylindrical radial coordinate [m]
R_0	electric resistance per unit length for stationary current [V A^{-1} m^{-1}]
s	dimensionless radial coordinate $s = r/b$
s', s''	integration variables
t	time [s]
T	temperature [K]
$\bar{T}$	time-averaged temperature [K]
T_f	fluid temperature outside the boundary layer [K]
u, v	dimensionless functions of s, Ω, Δ and Λ_c defined in equations (85) and (86)
w	Wronskian
x	real variable
Y_n	Bessel function of second kind and order n
z	cylindrical axial coordinate [m]
$\mathbf{z}$	unit vector parallel to the axis of the cylinder.

Greek symbols

Γ	dimensionless parameter $\Gamma = \lambda_d/\lambda_c$
Δ	dimensionless parameter $\Delta = a/b$
ε_0	electric permittivity of empty space [F m^{-1}]
θ	cylindrical angular coordinate, rad
ϑ	dimensionless temperature defined in equation (41)
$\vartheta_0, \vartheta_1, \vartheta_2$	dimensionless real functions defined in equations (44) and (60)
λ_c	thermal conductivity of the electric conductor [W m^{-1} K^{-1}]
λ_d	thermal conductivity of the dielectric solid [W m^{-1} K^{-1}]
Λ_c	dimensionless parameter $\Lambda_c = (2\omega\rho_c c_{p_c} b^2)^{1/2}/\lambda_c^{1/2}$
Λ_d	dimensionless parameter $\Lambda_d = (2\omega\rho_d c_{p_d} b^2)^{1/2}/\lambda_d^{1/2}$
μ_0	magnetic permeability of empty space [V s A^{-1} m^{-1}]
ρ_c	mass density of the electric conductor [kg m^{-3}]
ρ_d	mass density of the dielectric solid [kg m^{-3}]
σ	electric conductivity [A V^{-1} m^{-1}]
τ	dimensionless time $\tau = \omega t$
φ	dimensionless function of Ω and Δ defined in equation (94)
ψ	dimensionless complex-valued function, $\psi = \vartheta_1 + i\vartheta_2$
ω	angular frequency, rad s^{-1}
Ω	dimensionless parameter $\Omega = b\sqrt{\mu_0\omega\sigma}$.

Symbols

$\equiv$	equal by definition
$\lVert\ \rVert$	modulus of a complex number.

posed to transfer the power generated by the Joule effect to a surrounding fluid, through convection.

The aim of this paper is to analyse the steady periodic heat conduction in a cylindrical resistor with annular section and crossed by an alternating electric current. The hole within the cylinder is considered either as empty or as filled with a dielectric solid. The paper is organized as follows. First, the distribution of the power generated per unit volume within the resistor is evaluated. Then, the Fourier equation is written in a dimensionless form and the temperature field is determined analytically both in the domain occupied by the resistor and in the domain occupied by the dielectric as a function of position, time and proper dimensionless parameters. Moreover, the temperature field in the resistor when the hole within the cylinder is empty is obtained as a limiting case. Numerical values of the time-averaged and dimensionless temperature field and of the amplitude and phase of the dimensionless temperature oscillations are provided.

THE HEAT GENERATION

In this section, the power per unit volume generated by the Joule effect within an infinitely long cylindrical conductor with annular section carrying an alternating electric current is determined.

Let us consider an electric conductor in the form of a hollow cylinder, with infinite length, internal radius a and external radius b. The electric conductor has thermal conductivity λ_c and electric conductivity σ. A homogeneous dielectric solid with thermal conductivity λ_d occupies the region $r < a$. The magnetic permeability of the conductor and that of the dielectric solid will be assumed to be equal to μ_0, and the physical properties of both the electric conductor and the dielectric solid will be treated as constants. A parallel electric field $\mathbf{E}$ with direction $\mathbf{z}$ and a magnetic field $\mathbf{H}$, which depend only on r and t, are present inside the conductor. These fields undergo periodic variations in time, with an angular frequency ω. The macroscopic charge density distribution is zero everywhere. It will be assumed that the angular frequency ω of the electric field oscillations satisfies the conditions $\omega b \ll c$ and $\omega\varepsilon_0 \ll \sigma$, which are necessary for the validity of the quasi-stationary approximation of the electromagnetic field equations [7]. These conditions are satisfied even at very high frequencies. In fact, for an annular conductor with external radius $b = 1$ cm and electric conductivity $\sigma \simeq 10^7\ \mathrm{A\,V^{-1}\,m^{-1}}$, the condition $\omega b \ll c$ holds if $\omega \ll 10^{10}\ \mathrm{rad\,s^{-1}}$, while the condition $\omega\varepsilon_0 \ll \sigma$ holds if $\omega \ll 10^{18}\ \mathrm{rad\,s^{-1}}$. In the quasi-stationary approximation, for $a < r < b$, Maxwell's equations can be written as [7]

$$\nabla\cdot\mathbf{E} = 0 \tag{1}$$

$$\nabla\cdot\mathbf{H} = 0 \tag{2}$$

$$\nabla\times\mathbf{E} = -\mu_0\frac{\partial\mathbf{H}}{\partial t} \tag{3}$$

$$\nabla\times\mathbf{H} = \mathbf{J}. \tag{4}$$

Ohm's law is supposed to hold, i.e.

$$\mathbf{J} = \sigma\mathbf{E}. \tag{5}$$

For $r < a$, Maxwell's equations can be written as

$$\nabla\cdot\mathbf{E} = 0 \tag{6}$$

$$\nabla\cdot\mathbf{H} = 0 \tag{7}$$

$$\nabla\times\mathbf{E} = -\mu_0\frac{\partial\mathbf{H}}{\partial t} \tag{8}$$

$$\nabla\times\mathbf{H} = 0. \tag{9}$$

For every value of r, the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$ can be expressed as

$$\mathbf{E}(r,t) = \mathbf{E}_0(r)\,\mathrm{e}^{-\mathrm{i}\omega t},\quad \mathbf{H}(r,t) = \mathbf{H}_0(r)\,\mathrm{e}^{-\mathrm{i}\omega t}. \tag{10}$$

On account of equations (5) and (10), equations (1)–(4) yield

$$\nabla\cdot\mathbf{E}_0 = 0 \tag{11}$$

$$\nabla\cdot\mathbf{H}_0 = 0 \tag{12}$$

$$\nabla\times\mathbf{E}_0 = \mathrm{i}\mu_0\omega\mathbf{H}_0 \tag{13}$$

$$\nabla\times\mathbf{H}_0 = \sigma\mathbf{E}_0. \tag{14}$$

By applying the curl operator to both sides of equation (13) and by employing equation (14), one obtains

$$\nabla\times(\nabla\times\mathbf{E}_0) = \mathrm{i}\mu_0\omega\sigma\mathbf{E}_0. \tag{15}$$

As a consequence of the vector identity [9]

$$\nabla\times(\nabla\times\mathbf{E}_0) = \nabla(\nabla\cdot\mathbf{E}_0) - \nabla^2\mathbf{E}_0 \tag{16}$$

equation (15) can be rewritten as

$$\nabla(\nabla\cdot\mathbf{E}_0) - \nabla^2\mathbf{E}_0 = \mathrm{i}\mu_0\omega\sigma\mathbf{E}_0. \tag{17}$$

On account of equation (11), equation (17) yields

$$\nabla^2\mathbf{E}_0 + \mathrm{i}\mu_0\omega\sigma\mathbf{E}_0 = 0. \tag{18}$$

By projecting equation (18) along $\mathbf{z}$, one obtains

$$\frac{\mathrm{d}^2E_0}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}E_0}{\mathrm{d}r} + \mathrm{i}\mu_0\omega\sigma E_0 = 0. \tag{19}$$

The general solution of equation (19) is [10]

$$E_0 = AJ_0(r\sqrt{\mathrm{i}\mu_0\omega\sigma}) + BY_0(r\sqrt{\mathrm{i}\mu_0\omega\sigma}). \tag{20}$$

A constraint on the two integration constants A and B can be obtained as follows. For $r < a$, equations (6)–(10) yield

$$\nabla\cdot\mathbf{E}_0 = 0 \tag{21}$$

$$\nabla\cdot\mathbf{H}_0 = 0 \tag{22}$$

$$\nabla\times\mathbf{E}_0 = \mathrm{i}\mu_0\omega\mathbf{H}_0 \tag{23}$$

$$\nabla\times\mathbf{H}_0 = 0. \tag{24}$$

By projecting equation (24) along $\mathbf{z}$, one obtains [10]

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rH_{0\theta}) = 0,\quad \text{for } r < a. \tag{25}$$

On account of equation (25), since $H_{0\theta}$ cannot be singular at $r = 0$, one has $H_{0\theta} = 0$ for $r < a$. Therefore, the projection of equation (23) along the direction θ yields

$$-\frac{\mathrm{d}E_{0z}}{\mathrm{d}r} = \mathrm{i}\mu_0\omega H_{0\theta} = 0, \quad \text{for } r < a. \tag{26}$$

Moreover, the projection of equation (13) along the direction θ yields

$$-\frac{\mathrm{d}E_0}{\mathrm{d}r} = \mathrm{i}\mu_0\omega H_{0\theta}, \quad \text{for } a < r < b. \tag{27}$$

Since $\mathbf{J}$ is not singular at $r = a$, i.e. no surface charge density distribution is present at $r = a$, then $H_{0\theta}$ must be continuous at $r = a$ [7]. As a consequence of equations (26)–(27), also $\mathrm{d}E_{0z}/\mathrm{d}r$ is continuous at $r = a$ and equals zero. Therefore, on account of the identities [9]

$$\frac{\mathrm{d}J_0(x)}{\mathrm{d}x} = -J_1(x) \tag{28}$$

$$\frac{\mathrm{d}Y_0(x)}{\mathrm{d}x} = -Y_1(x) \tag{29}$$

and of equation (20) one obtains

$$0 = \left.\frac{\mathrm{d}E_0}{\mathrm{d}r}\right|_{r=a} = -\sqrt{\mathrm{i}\mu_0\omega\sigma}[AJ_1(a\sqrt{\mathrm{i}\mu_0\omega\sigma}) + BY_1(a\sqrt{\mathrm{i}\mu_0\omega\sigma})]. \tag{30}$$

On account of equation (30), equation (20) can be rewritten as

$$E_0 = A\left[J_0(r\sqrt{\mathrm{i}\mu_0\omega\sigma}) - \frac{J_1(a\sqrt{\mathrm{i}\mu_0\omega\sigma})}{Y_1(a\sqrt{\mathrm{i}\mu_0\omega\sigma})} Y_0(r\sqrt{\mathrm{i}\mu_0\omega\sigma})\right]. \tag{31}$$

If one defines the dimensionless parameters $\Omega = b\sqrt{\mu_0\omega\sigma}$ and $\Delta = a/b$, the dimensionless coordinate $s = r/b$ and the dimensionless function

$$f(s,\Omega,\Delta) \equiv J_0(\sqrt{\mathrm{i}}\Omega s) - \frac{J_1(\sqrt{\mathrm{i}}\Omega\Delta)}{Y_1(\sqrt{\mathrm{i}}\Omega\Delta)} Y_0(\sqrt{\mathrm{i}}\Omega s) \tag{32}$$

equation (31) can be rewritten as

$$E_0 = Af(s,\Omega,\Delta). \tag{33}$$

The integration constant A can be evaluated by the condition that the time-averaged power generated per unit length within the annulus has a prescribed value Q. The power generated per unit volume within the annulus can be evaluated as $q_g = \mathbf{J}\cdot\mathbf{E}$ [11]. Therefore, on account of equations (5), (10) and (33), q_g is given by

$$\begin{aligned} q_g &= \sigma[\mathrm{Re}(E_0\,\mathrm{e}^{-\mathrm{i}\omega t})]^2 \\ &= \frac{\sigma}{2}[\|E_0\|^2 + \mathrm{Re}(E_0^2)\cos(2\omega t) + \mathrm{Im}(E_0^2)\sin(2\omega t)] \\ &= \frac{\sigma}{2}A^2\{\|f(s,\Omega,\Delta)\|^2 + \mathrm{Re}[f(s,\Omega,\Delta)^2]\cos(2\omega t) \\ &\quad + \mathrm{Im}[f(s,\Omega,\Delta)^2]\sin(2\omega t)\}. \end{aligned} \tag{34}$$

On account of equation (34), the time average of q_g can be evaluated as

$$\overline{q_g} = \frac{\omega}{\pi}\int_0^{\pi/\omega} q_g\,\mathrm{d}t = \frac{\sigma}{2}A^2\|f(s,\Omega,\Delta)\|^2. \tag{35}$$

Therefore, the time-averaged power generated per unit length within the annulus is given by

$$Q = \int_a^b \overline{q_g}\,2\pi r\,\mathrm{d}r = \sigma\pi b^2A^2\int_\Delta^1 \|f(s',\Omega,\Delta)\|^2 s'\,\mathrm{d}s' \tag{36}$$

so that A^2 can be expressed as

$$A^2 = \frac{Q}{\sigma\pi b^2\int_\Delta^1 \|f(s',\Omega,\Delta)\|^2 s'\,\mathrm{d}s'}. \tag{37}$$

Let us define the dimensionless function

$$g(s,\Omega,\Delta) \equiv \frac{f(s,\Omega,\Delta)^2}{\int_\Delta^1 \|f(s',\Omega,\Delta)\|^2 s'\,\mathrm{d}s'}. \tag{38}$$

On account of equations (37) and (38), equation (34) can be rewritten as

$$q_g = \frac{Q}{2\pi b^2}\{\|g(s,\Omega,\Delta)\| + \mathrm{Re}[g(s,\Omega,\Delta)]\cos(2\omega t) + \mathrm{Im}[g(s,\Omega,\Delta)]\sin(2\omega t)\}. \tag{39}$$

The expression of q_g for a solid cylinder can be easily obtained from equations (38) and (39) in the limit $\Delta \to 0$. In this limit, on account of equation (32), $f(s,\Omega,\Delta) \to J_0(\sqrt{\mathrm{i}}\Omega s)$, in agreement with the result obtained in [6].

DIMENSIONLESS FORM OF THE FOURIER EQUATION FOR THE DIELECTRIC SOLID

In this section, the Fourier equation for the region $r < a$ is written in a dimensionless form. Then, under the hypothesis that the heat conduction is steady periodic, the dimensionless Fourier equation is transformed into a system of three ordinary differential equations and the general solutions of these equations are determined analytically.

The Fourier equation for the dielectric solid can be written as

$$\lambda_d\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) = \rho_d c_{p_d}\frac{\partial T}{\partial t}. \tag{40}$$

If a dimensionless temperature

$$\vartheta \equiv \frac{2\pi\lambda_c}{Q}(T - T_f) \tag{41}$$

is defined and if the dimensionless radius s and the dimensionless time $\tau = \omega t$ are employed, then equation (40) can be rewritten as

$$\frac{\partial^2 \vartheta}{\partial s^2}+\frac{1}{s}\frac{\partial \vartheta}{\partial s}=\frac{\omega \rho_d c_{p_d} b^2}{\lambda_d}\frac{\partial \vartheta}{\partial \tau}. \quad (42)$$

By employing the dimensionless parameter $\Lambda_d = (2\omega \rho_d c_{p_d} b^2)^{1/2}/\lambda_d^{1/2}$, equation (42) yields

$$\frac{\partial^2 \vartheta}{\partial s^2}+\frac{1}{s}\frac{\partial \vartheta}{\partial s}=\frac{\Lambda_d^2}{2}\frac{\partial \vartheta}{\partial \tau}. \quad (43)$$

After a time sufficiently long for transient terms to be neglected, the temperature field becomes steady periodic, so that

$$\vartheta(s,\tau) = \vartheta_0(s)+\vartheta_1(s)\cos(2\tau)+\vartheta_2(s)\sin(2\tau). \quad (44)$$

By substituting equation (44) into equation (43), one obtains

$$\frac{d^2\vartheta_0}{ds^2}+\frac{1}{s}\frac{d\vartheta_0}{ds}+\left\{\frac{d^2\vartheta_1}{ds^2}+\frac{1}{s}\frac{d\vartheta_1}{ds}-\Lambda_d^2\vartheta_2\right\}\cos(2\tau)$$
$$+\left\{\frac{d^2\vartheta_2}{ds^2}+\frac{1}{s}\frac{d\vartheta_2}{ds}+\Lambda_d^2\vartheta_1\right\}\sin(2\tau)=0. \quad (45)$$

The integration of both sides of equation (45) with respect to τ in the interval $[0,\pi]$ yields

$$\frac{d^2\vartheta_0}{ds^2}+\frac{1}{s}\frac{d\vartheta_0}{ds}=0. \quad (46)$$

If one multiplies both sides of equation (45) by $\cos(2\tau)$, the integration with respect to τ in the interval $[0,\pi]$ yields

$$\frac{d^2\vartheta_1}{ds^2}+\frac{1}{s}\frac{d\vartheta_1}{ds}-\Lambda_d^2\vartheta_2=0. \quad (47)$$

If one multiplies both sides of equation (45) by $\sin(2\tau)$, the integration with respect to τ in the interval $[0,\pi]$ yields

$$\frac{d^2\vartheta_2}{ds^2}+\frac{1}{s}\frac{d\vartheta_2}{ds}+\Lambda_d^2\vartheta_1=0. \quad (48)$$

By employing the complex valued function $\psi = \vartheta_1+i\vartheta_2$ equations (47) and (48) collapse into one differential equation, namely

$$\frac{d^2\psi}{ds^2}+\frac{1}{s}\frac{d\psi}{ds}+i\Lambda_d^2\psi=0. \quad (49)$$

Note that ϑ_0 represents the time-average of ϑ, while the modulus and the argument of ψ represent, respectively, the amplitude and the phase of the dimensionless temperature oscillations.

The general solution of equation (46) is

$$\vartheta_0 = c_1\ln(s)+c_2. \quad (50)$$

Since ϑ_0 cannot be singular for $s = 0$, the integration constant c_1 must be zero and equation (50) yields

$$\vartheta_0 = c_2. \quad (51)$$

The general solution of equation (49) is

$$\psi = c_3 J_0(\sqrt{i}\Lambda_d s)+c_4 Y_0(\sqrt{i}\Lambda_d s). \quad (52)$$

Since ψ cannot be singular, while $Y_0(\sqrt{i}\Lambda_d s)$ is singular for $s = 0$, then c_4 must be zero and equation (52) yields

$$\psi = c_3 J_0(\sqrt{i}\Lambda_d s). \quad (53)$$

DIMENSIONLESS FORM OF THE FOURIER EQUATION FOR THE CONDUCTOR

In this section, the Fourier equation for the region $a < r < b$ is written in a dimensionless form. Then, under the hypothesis that the heat conduction is steady periodic, the dimensionless Fourier equation is transformed into a system of two ordinary differential equations.

The Fourier equation for the annular conductor can be written as

$$\lambda_c\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}\right)+q_g = \rho_c c_{p_c}\frac{\partial T}{\partial t}. \quad (54)$$

Let convection be present at the surface $r = b$ with an external fluid which has temperature T_f outside the boundary layer. Then, the boundary condition on function T at $r = b$ can be expressed as

$$-\lambda_c\left.\frac{\partial T}{\partial r}\right|_{r=b} = h[T(b,t)-T_f] \quad (55)$$

and Q is related to T_f by

$$Q = 2\pi bh[\bar{T}(b)-T_f]. \quad (56)$$

The convection coefficient h is supposed to be independent of temperature; this hypothesis is verified with an excellent approximation in the case of forced convection. By employing the dimensionless temperature ϑ, the dimensionless radius s and the dimensionless time τ, equation (54) can be rewritten as

$$\frac{\partial^2 \vartheta}{\partial s^2}+\frac{1}{s}\frac{\partial \vartheta}{\partial s}+\frac{2\pi b^2}{Q}q_g=\frac{\omega \rho_c c_{p_c} b^2}{\lambda_c}\frac{\partial \vartheta}{\partial \tau}. \quad (57)$$

As a consequence of equation (39) and of the definition of the dimensionless parameter $\Lambda_c = (2\omega\rho_c c_{p_c} b^2)^{1/2}/\lambda_c^{1/2}$, equation (57) yields

$$\frac{\partial^2 \vartheta}{\partial s^2}+\frac{1}{s}\frac{\partial \vartheta}{\partial s}+\|g(s,\Omega,\Delta)\|+\mathrm{Re}[g(s,\Omega,\Delta)]\cos(2\tau)$$
$$+\mathrm{Im}[g(s,\Omega,\Delta)]\sin(2\tau)=\frac{\Lambda_c^2}{2}\frac{\partial \vartheta}{\partial \tau}. \quad (58)$$

By employing the dimensionless radius s, the dimensionless time τ, the dimensionless temperature ϑ and the Biot number $Bi = hb/\lambda_c$, equation (55) can be written as

$$\left.\frac{\partial \vartheta}{\partial s}\right|_{s=1}+Bi\,\vartheta(1,\tau)=0. \quad (59)$$

After a time sufficiently long for transient terms to be neglected, the temperature field becomes steady periodic, so that

$$\vartheta(s,\tau) = \vartheta_0(s) + \vartheta_1(s)\cos(2\tau) + \vartheta_2(s)\sin(2\tau). \tag{60}$$

By substituting equation (60) in equation (58), one obtains

$$\frac{d^2\vartheta_0}{ds^2} + \frac{1}{s}\frac{d\vartheta_0}{ds} + \| g(s,\Omega,\Delta)\| + \left\{\frac{d^2\vartheta_1}{ds^2} + \frac{1}{s}\frac{d\vartheta_1}{ds} + \mathrm{Re}\,[g(s,\Omega,\Delta)] - \Lambda_c^2\vartheta_2\right\}\cos(2\tau) + \left\{\frac{d^2\vartheta_2}{ds^2} + \frac{1}{s}\frac{d\vartheta_2}{ds} + \mathrm{Im}\,[g(s,\Omega,\Delta)] + \Lambda_c^2\vartheta_1\right\}\sin(2\tau) = 0. \tag{61}$$

The integration of both sides of equation (61) with respect to τ in the interval $[0,\pi]$ yields

$$\frac{d^2\vartheta_0}{ds^2} + \frac{1}{s}\frac{d\vartheta_0}{ds} = -\| g(s,\Omega,\Delta)\|. \tag{62}$$

If one multiplies both sides of equation (61) by $\cos(2\tau)$, the integration with respect to τ in the interval $[0,\pi]$ yields

$$\frac{d^2\vartheta_1}{ds^2} + \frac{1}{s}\frac{d\vartheta_1}{ds} - \Lambda_c^2\vartheta_2 = -\mathrm{Re}\,[g(s,\Omega,\Delta)]. \tag{63}$$

If one multiplies both sides of equation (61) by $\sin(2\tau)$, the integration with respect to τ in the interval $[0,\pi]$ yields

$$\frac{d^2\vartheta_2}{ds^2} + \frac{1}{s}\frac{d\vartheta_2}{ds} + \Lambda_c^2\vartheta_1 = -\mathrm{Im}\,[g(s,\Omega,\Delta)]. \tag{64}$$

By employing the complex valued quantity $\psi = \vartheta_1 + i\vartheta_2$ equations (63) and (64) collapse into one differential equation, namely

$$\frac{d^2\psi}{ds^2} + \frac{1}{s}\frac{d\psi}{ds} + i\Lambda_c^2\psi = -g(s,\Omega,\Delta). \tag{65}$$

By substituting equation (60) in equation (59) and by employing the same method as that used to split equation (61) into equations (62) and (65), one obtains

$$\left.\frac{d\vartheta_0}{ds}\right|_{s=1} + Bi\,\vartheta_0(1) = 0 \tag{66}$$

$$\left.\frac{d\psi}{ds}\right|_{s=1} + Bi\,\psi(1) = 0. \tag{67}$$

The boundary conditions (66) and (67) are not sufficient to determine uniquely ϑ_0 and ψ in the region $\Delta < s < 1$; another set of two boundary conditions, one for ϑ_0 and one for ψ, is necessary and can be determined as follows. On account of equations (51) and (53) and of identity (28), functions ϑ_0 and ψ in the region $\Delta < s < 1$ must fulfil the following matching conditions

$$\vartheta_0(\Delta) = c_2 \tag{68}$$

$$-\lambda_c \left.\frac{d\vartheta_0}{ds}\right|_{s=\Delta} = 0 \tag{69}$$

$$\psi(\Delta) = c_3 J_0(\sqrt{i}\Lambda_d\Delta) \tag{70}$$

$$-\lambda_c \left.\frac{d\psi}{ds}\right|_{s=\Delta} = \lambda_d c_3 \sqrt{i}\Lambda_d J_1(\sqrt{i}\Lambda_d\Delta). \tag{71}$$

Equations (68) and (70) are a consequence of the continuity of the temperature field, while equations (69) and (71) are due to the continuity of the heat flux vector. By employing the dimensionless parameter $\Gamma = \lambda_d/\lambda_c$, equations (70) and (71) yield

$$\frac{1}{\psi(\Delta)}\left.\frac{d\psi}{ds}\right|_{s=\Delta} = -\sqrt{i}\Gamma\Lambda_d \frac{J_1(\sqrt{i}\Lambda_d\Delta)}{J_0(\sqrt{i}\Lambda_d\Delta)}. \tag{72}$$

Equations (69) and (72) represent the missing boundary conditions for ϑ_0 and ψ necessary to determine uniquely these functions in the region $\Delta < s < 1$.

EVALUATION OF THE DIMENSIONLESS TEMPERATURE FIELD

In this section, the temperature field in the whole region $r \leqslant b$ is determined. First, the general solutions of equations (62) and (65) are evaluated. Then, the boundary conditions (66) and (67) together with the matching conditions (69) and (72) are employed to determine the temperature field.

The general solution of equation (62) is easily obtained as follows. Equation (62) can be rewritten as

$$\frac{d}{ds}\left(s\frac{d\vartheta_0}{ds}\right) = -s\| g(s,\Omega,\Delta)\|. \tag{73}$$

A first integration of equation (73) yields

$$s\frac{d\vartheta_0}{ds} = -\int_\Delta^s s'\| g(s',\Omega,\Delta)\|\,ds' + \Delta \left.\frac{d\vartheta}{ds}\right|_{s=\Delta}. \tag{74}$$

On account of equation (69), equation (74) can be simplified as follows:

$$s\frac{d\vartheta_0}{ds} = -\int_\Delta^s s'\| g(s',\Omega,\Delta)\|\,ds'. \tag{75}$$

The integration of equation (75) leads to

$$\vartheta_0(s) = \vartheta_0(1) + \int_s^1 \frac{1}{s''}\left[\int_\Delta^{s''} s'\| g(s',\Omega,\Delta)\|\,ds'\right] ds''. \tag{76}$$

On account of equation (38), equation (75) yields $d\vartheta_0/ds|_{s=1} = -1$. Therefore, as a consequence of equation (66), one obtains $\vartheta_0(1) = 1/Bi$ and equation (76) can be rewritten as

$$\vartheta_0(s) = \frac{1}{Bi} + \int_s^1 \frac{1}{s''}\left[\int_\Delta^{s''} s'\| g(s',\Omega,\Delta)\|\,ds'\right] ds''. \tag{77}$$

Table 1. Values of $\vartheta_0(s)-1/Bi$ for $0 \leqslant \Omega \leqslant 20$, $\Delta = 0.4$, $\Delta = 0.7$

	$\Delta = 0.4$			$\Delta = 0.7$		
Ω	$s = 0.4$	$s = 0.6$	$s = 0.8$	$s = 0.7$	$s = 0.8$	$s = 0.9$
0	0.3255	0.2837	0.1718	0.1573	0.1385	0.0850
1	0.3248	0.2831	0.1716	0.1573	0.1385	0.0850
2	0.3153	0.2754	0.1683	0.1569	0.1382	0.0849
3	0.2818	0.2483	0.1566	0.1553	0.1369	0.0843
4	0.2270	0.2038	0.1373	0.1512	0.1336	0.0829
5	0.1750	0.1614	0.1185	0.1436	0.1272	0.0801
6	0.1384	0.1310	0.1042	0.1322	0.1178	0.0759
7	0.1145	0.1107	0.0935	0.1182	0.1063	0.0708
8	0.0982	0.0962	0.0848	0.1037	0.0944	0.0655
9	0.0862	0.0851	0.0774	0.0905	0.0834	0.0605
10	0.0768	0.0762	0.0709	0.0794	0.0742	0.0563
11	0.0692	0.0689	0.0653	0.0705	0.0667	0.0527
12	0.0630	0.0629	0.0603	0.0635	0.0607	0.0497
13	0.0578	0.0577	0.0559	0.0578	0.0559	0.0471
14	0.0534	0.0534	0.0521	0.0533	0.0519	0.0448
15	0.0496	0.0496	0.0487	0.0495	0.0485	0.0427
16	0.0464	0.0464	0.0457	0.0462	0.0455	0.0408
17	0.0435	0.0435	0.0431	0.0434	0.0429	0.0390
18	0.0410	0.0410	0.0407	0.0409	0.0406	0.0374
19	0.0387	0.0387	0.0385	0.0387	0.0385	0.0358
20	0.0367	0.0367	0.0366	0.0367	0.0365	0.0343

In the limit $\Delta \to 0$, equation (77) agrees with the expression of the time-averaged and dimensionless temperature field obtained for a solid cylinder in [6]. When $\Omega \to 0$, on account of equations (32) and (38), equation (77) yields

$$\vartheta_0(s) = \frac{1}{Bi} + \frac{2\Delta^2 \ln s + 1 - s^2}{2(1-\Delta^2)}. \tag{78}$$

By employing equations (32) and (38) and by evaluating numerically the integrals in equation (77), one obtains $\vartheta_0(s)$. Values of $\vartheta_0(s)-1/Bi$ for $0 \leqslant \Omega \leqslant 20$, $\Delta = 0.4$ and $\Delta = 0.7$ are reported in Table 1. Note that, on account of equations (51) and (68), $\vartheta_0(s)$ is uniform in the region $0 \leqslant s \leqslant \Delta$ and has the value $\vartheta_0(\Delta)$, which can be determined by Table 1. Moreover, this table shows that, as Ω increases, $\vartheta_0(s)$ tends to become uniformly distributed in the whole cylinder.

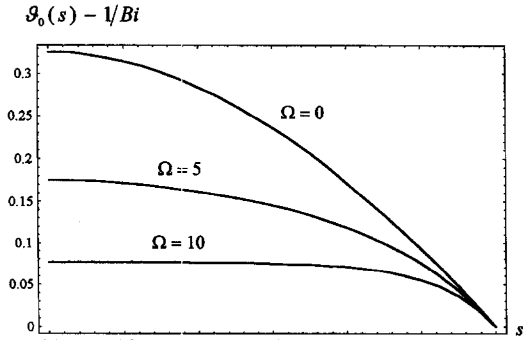

Fig. 1. Plots of $\vartheta_0(s)-1/Bi$ for $\Delta = 0.4$ and three values of Ω.

In fact, as a consequence of equations (32), (38) and (77), when $\Omega \to \infty$ function $\vartheta_0(s)$ tends to $1/Bi$ for every $s \leqslant 1$. In Fig. 1, the time-averaged and dimensionless temperature $\vartheta_0(s)$ is plotted as a function of s in the interval $\Delta \leqslant s \leqslant 1$, for $\Delta = 0.4$. Three curves are reported, for $\Omega = 0$, $\Omega = 5$ and $\Omega = 10$.

Let us now solve equation (65) with the boundary conditions (67) and (72). The homogeneous differential equation associated with equation (65) is

$$\frac{d^2\psi}{ds^2} + \frac{1}{s}\frac{d\psi}{ds} + i\Lambda_c^2\psi = 0. \tag{79}$$

Two linearly independent solutions of equation (79) are $J_0(\sqrt{i}\Lambda_c s)$ and $Y_0(\sqrt{i}\Lambda_c s)$ [10]. The Wronskian associated with this pair of solutions is

$$w(s) = J_0(\sqrt{i}\Lambda_c s)\frac{d}{ds}Y_0(\sqrt{i}\Lambda_c s) - Y_0(\sqrt{i}\Lambda_c s)\frac{d}{ds}J_0(\sqrt{i}\Lambda_c s). \tag{80}$$

On account of the identity [9]

$$J_0(x)\frac{d}{dx}Y_0(x) - Y_0(x)\frac{d}{dx}J_0(x) = \frac{2}{\pi x} \tag{81}$$

equation (80) yields

$$w(s) = \frac{2}{\pi s}. \tag{82}$$

By employing the method of variation of parameters presented in [12], the general solution of equation (65) can be written as

$$\psi(s) = k_1 J_0(\sqrt{i}\Lambda_c s) + k_2 Y_0(\sqrt{i}\Lambda_c s) - Y_0(\sqrt{i}\Lambda_c s)\int_\Delta^s \frac{J_0(\sqrt{i}\Lambda_c s')}{w(s')} g(s', \Omega, \Delta)\, ds' + J_0(\sqrt{i}\Lambda_c s)\int_\Delta^s \frac{Y_0(\sqrt{i}\Lambda_c s')}{w(s')} g(s', \Omega, \Delta)\, ds' \quad (83)$$

i.e. on account of equation (82)

$$\psi(s) = k_1 J_0(\sqrt{i}\Lambda_c s) + k_2 Y_0(\sqrt{i}\Lambda_c s) + \frac{\pi}{2}\int_\Delta^s [J_0(\sqrt{i}\Lambda_c s) Y_0(\sqrt{i}\Lambda_c s') - J_0(\sqrt{i}\Lambda_c s') Y_0(\sqrt{i}\Lambda_c s)] g(s', \Omega, \Delta) s'\, ds'. \quad (84)$$

Let us define functions $u(s, \Omega, \Delta, \Lambda_c)$ and $v(s, \Omega, \Delta, \Lambda_c)$ as follows:

$$u(s, \Omega, \Delta, \Lambda_c) \equiv \int_\Delta^s [J_0(\sqrt{i}\Lambda_c s) Y_0(\sqrt{i}\Lambda_c s') - J_0(\sqrt{i}\Lambda_c s') Y_0(\sqrt{i}\Lambda_c s)] g(s', \Omega, \Delta) s'\, ds' \quad (85)$$

$$v(s, \Omega, \Delta, \Lambda_c) \equiv \int_\Delta^s [J_1(\sqrt{i}\Lambda_c s) Y_0(\sqrt{i}\Lambda_c s') - J_0(\sqrt{i}\Lambda_c s') Y_1(\sqrt{i}\Lambda_c s)] g(s', \Omega, \Delta) s'\, ds'. \quad (86)$$

On account of equation (85), equation (84) can be rewritten as

$$\psi(s) = k_1 J_0(\sqrt{i}\Lambda_c s) + k_2 Y_0(\sqrt{i}\Lambda_c s) + \frac{\pi}{2} u(s, \Omega, \Delta, \Lambda_c). \quad (87)$$

By substituting equation (87) into equations (67) and (72), and by employing equations (28), (29), (85) and (86), one can determine the integration constants k_1 and k_2, namely

$$k_1 = \frac{\pi}{2}[\sqrt{i}\Lambda_c v(1, \Omega, \Delta, \Lambda_c) - Bi\, u(1, \Omega, \Delta, \Lambda_c)] \times \{Bi[J_0(\sqrt{i}\Lambda_c) - C Y_0(\sqrt{i}\Lambda_c)] - \sqrt{i}\Lambda_c[J_1(\sqrt{i}\Lambda_c) - C Y_1(\sqrt{i}\Lambda_c)]\}^{-1} \quad (88)$$

$$k_2 = -C k_1 \quad (89)$$

where the dimensionless constant C is given by

$$C \equiv [\Gamma\Lambda_d J_1(\sqrt{i}\Lambda_d\Delta) J_0(\sqrt{i}\Lambda_c\Delta) - \Lambda_c J_1(\sqrt{i}\Lambda_c\Delta) \times J_0(\sqrt{i}\Lambda_d\Delta)][\Gamma\Lambda_d J_1(\sqrt{i}\Lambda_d\Delta) Y_0(\sqrt{i}\Lambda_c\Delta) - \Lambda_c Y_1(\sqrt{i}\Lambda_c\Delta) J_0(\sqrt{i}\Lambda_d\Delta)]^{-1}. \quad (90)$$

Values of the modulus and of the argument of $Bi\,\psi$, determined by equations (87), (53) and (70), are reported in Tables 2, 3 and 4 at $s = 1$, $s = \Delta$ and $s = 0$ respectively. In these tables a tungsten conductor with $\lambda_c = 163.0$ W m^{-1} K^{-1}, $\rho_c = 1.925 \times 10^4$ kg m^{-3}, $c_{p_c} = 135.0$ J kg^{-1} K^{-1} and $\sigma = 1.370 \times 10^7$ A V^{-1} m^{-1} is considered. The region $0 \leqslant r \leqslant a$ is occupied either by a Pyrex glass with $\lambda_d = 1.45$ W m^{-1} K^{-1} and $\lambda_d/(\rho_d c_{p_d}) = 0.740 \times 10^{-6}$ m^2 s^{-1}, or by Teflon with $\lambda_d = 0.260$ W m^{-1} K^{-1} and $\lambda_d/(\rho_d c_{p_d}) = 0.340 \times 10^{-6}$ m^2 s^{-1}. With these choices, Λ_c/Ω is equal to 43.03, Γ is equal either to 0.00890 for Pyrex or to 0.00159 for Teflon, Λ_d/Ω is equal either to 396.2 for Pyrex or to 584.4 for Teflon. The value of Δ is chosen as 0.4, while Bi can assume the values 10^{-3}, 10^{-2} or 10^{-1}.

If in the region $r \leqslant a$ the dielectric solid is replaced by empty space, the surface $r = a$ becomes adiabatic. In this case, equations (85)–(90) yield the expression for ψ provided that, in equation (90), the limit $\Gamma \to 0$ is performed. In fact, provided that $\Lambda_d < \infty$, when $\Gamma \to 0$ equation (72) yields $d\psi/ds|_{s=\Delta} = 0$, so that, on account of equations (60) and (69) one obtains $\partial\vartheta/\partial s|_{s=\Delta} = 0$. In the limit $\Gamma \to 0$, equation (90) yields

$$C = \frac{J_1(\sqrt{i}\,\Lambda_c\Delta)}{Y_1(\sqrt{i}\,\Lambda_c\Delta)} \quad (91)$$

so that, on account of equations (32) and (89), equation (87) can be rewritten as

$$\psi(s) = k_1 f(s, \Lambda_c, \Delta) + \frac{\pi}{2} u(s, \Omega, \Delta, \Lambda_c). \quad (92)$$

Values of the modulus and of the phase of $Bi\,\psi$, evaluated by equation (92) for the tungsten conductor described above, are reported in Table 5 for $s = 1$ and in Table 6 for $s = \Delta$. The dimensionless parameter Δ can assume the values 0.4 and 0.7, while Bi can assume the values 10^{-3}, 10^{-2} or 10^{-1}.

DISCUSSION OF THE RESULTS

In the previous sections, an analytical solution of the heat conduction problem has been provided for an infinitely long solid cylinder with radius b and such that the region $0 \leqslant r \leqslant a$ is occupied by a dielectric solid, while the region $a \leqslant r \leqslant b$ is occupied by an electric conductor crossed by an alternating current. The heat generation due to the Joule effect is present only in the region $a \leqslant r \leqslant b$. The analytical solution of the Fourier equation has been obtained in a dimensionless form in a steady periodic regime, in the presence of convective heat transfer with an external fluid. The dimensionless temperature field depends on six dimensionless parameters: Bi, Ω, Δ, Λ_c, Γ and Λ_d. The time-averaged and dimensionless temperature distribution ϑ_0 depends on Bi and Ω, while $\vartheta_0 - 1/Bi$ depends only on Ω. Moreover, in the region $0 \leqslant r \leqslant a$, ϑ_0 is uniform. The values of $\vartheta_0 - 1/Bi$ reported in Table

Table 2. Values of the modulus and of the argument (in italic) of $Bi\psi$ at $s = 1$ for a tungsten conductor with $\Delta = 0.4$ and the region $0 \leqslant s \leqslant 0.4$ occupied either by a Pyrex glass or by Teflon

	$\Gamma = 0.00890$; $\Lambda_d/\Omega = 396.2$ (Pyrex glass)			$\Gamma = 0.00159$; $\Lambda_d/\Omega = 584.4$ (Teflon)		
Ω	$Bi = 10^{-3}$	$Bi = 10^{-2}$	$Bi = 10^{-1}$	$Bi = 10^{-3}$	$Bi = 10^{-2}$	$Bi = 10^{-1}$
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	0.0000	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0000*
0.01	0.0114	0.1126	0.7322	0.0124	0.1225	0.7646
	1.5236	*1.4221*	*0.7137*	*1.5377*	*1.4273*	*0.6796*
0.02	0.0030	0.0301	0.2800	0.0032	0.0315	0.2952
	1.5174	*1.4903*	*1.2360*	*1.5525*	*1.5241*	*1.2555*
0.03	0.0014	0.0137	0.1330	0.0014	0.0141	0.1373
	1.5299	*1.5175*	*1.3968*	*1.5575*	*1.5448*	*1.4204*
0.04	0.0008	0.0078	0.0764	0.0008	0.0080	0.0780
	1.5364	*1.5292*	*1.4590*	*1.5596*	*1.5523*	*1.4810*
0.05	0.0005	0.0050	0.0495	0.0005	0.0051	0.0502
	1.5402	*1.5355*	*1.4890*	*1.5604*	*1.5557*	*1.5088*
0.06	0.0004	0.0035	0.0346	0.0004	0.0036	0.0349
	1.5428	*1.5394*	*1.5056*	*1.5607*	*1.5572*	*1.5234*
0.07	0.0003	0.0026	0.0256	0.0003	0.0026	0.0257
	1.5448	*1.5421*	*1.5158*	*1.5606*	*1.5579*	*1.5317*
0.08	0.0002	0.0020	0.0197	0.0002	0.0020	0.0197
	1.5465	*1.5443*	*1.5226*	*1.5603*	*1.5581*	*1.5366*
0.09	0.0002	0.0016	0.0156	0.0002	0.0016	0.0156
	1.5481	*1.5462*	*1.5276*	*1.5599*	*1.5580*	*1.5395*
0.10	0.0001	0.0013	0.0127	0.0001	0.0013	0.0127
	1.5495	*1.5478*	*1.5313*	*1.5593*	*1.5576*	*1.5412*
0.20	0.0000	0.0003	0.0032	0.0000	0.0003	0.0032
	—	*1.5428*	*1.5348*	—	*1.5420*	*1.5340*
0.40	0.0000	0.0001	0.0008	0.0000	0.0001	0.0008
	—	*1.4701*	*1.4662*	—	*1.4701*	*1.4663*
0.60	0.0000	0.0000	0.0002	0.0000	0.0000	0.0002
	—	—	*1.3124*	—	—	*1.3124*

1 and those plotted in Fig. 1 show how ϑ_0 tends to become uniform throughout the cylinder and equal to $1/Bi$ as Ω tends to infinity.

The amplitude and the phase of the dimensionless temperature oscillations have been reported in Tables 2, 3 and 4, for prescribed values of Δ and of Λ_c/Ω respectively for $r = b$, $r = a$ and $r = 0$. Note that Λ_c/Ω depends only on the choice of the electric conductor, while Λ_d/Ω and Γ depend on the choice of both the dielectric solid and the electric conductor. Tables 2 and 3 show that both at $r = b$ and at $r = a$ the amplitude of the dimensionless temperature oscillations tends to decrease as Ω increases and becomes negligible for $\Omega \geqslant 0.6$. Moreover, Table 4 shows that, as Ω increases, the dimensionless temperature oscillations in the dielectric at $r = 0$ are strongly damped both in the case of Pyrex glass and in the case of Teflon, and have a negligible amplitude even at $\Omega = 0.06$. For the tungsten conductor considered in Tables 2–4, if $b = 1$ cm, the value $\Omega = 0.06$ corresponds to an electric current frequency of 0.333 Hz, while the value $\Omega = 0.6$ corresponds to a frequency of 33.3 Hz. For the values of Λ_d/Ω and Γ that correspond to Teflon, the amplitude of the dimensionless temperature oscillations at $r = a$ is slightly greater than at $r = b$. The phase difference of the dimensionless temperature oscillations between the surfaces $r = b$ and $r = a$ is very small if compared with that between the axis of the cylinder and the surface $r = a$, and is smaller for Teflon than for Pyrex.

The amplitude and the phase of the dimensionless temperature oscillations when the surface $r = a$ is adiabatic are reported in Table 5 for $r = b$ and in Table 6 for $r = a$. In these tables, the value of Λ_c/Ω is fixed and corresponds to a tungsten conductor. The results obtained in Table 5 for $\Delta = 0.4$ are very similar to those obtained in Table 2 for Teflon; the same analogy holds between Tables 6 and 3. In fact, the value of Γ for Teflon is so small that the effect of this dielectric on the dimensionless temperature distribution within the conductor is similar to that of vacuum. An analysis of Tables 5 and 6 shows that the values of the amplitude and of the phase at $r = b$ and at $r = a$ are very similar. In particular, the amplitude at $r = a$ is slightly greater than the amplitude at $r = b$.

Tables 2–6 show that the decrease in amplitude of the dimensionless temperature oscillations is strongly influenced by Bi: the smaller is Bi the smaller is the value of Ω above which the amplitude becomes negligible. Moreover, Tables 2–6 point out that, when $\Omega \to 0$, the amplitude of the dimensionless temperature oscillations tends to coincide with the time-

Table 3. Values of the modulus and of the argument (in italic) of $Bi\psi$ at $s = \Delta$ for a tungsten conductor with $\Delta = 0.4$ and the region $0 \leqslant s \leqslant 0.4$ occupied either by a Pyrex glass or by Teflon

	$\Gamma = 0.00890$; $\Lambda_d/\Omega = 396.2$ (Pyrex glass)			$\Gamma = 0.00159$; $\Lambda_d/\Omega = 584.4$ (Teflon)		
Ω	$Bi = 10^{-3}$	$Bi = 10^{-2}$	$Bi = 10^{-1}$	$Bi = 10^{-3}$	$Bi = 10^{-2}$	$Bi = 10^{-1}$
0.00	1.0003	1.0033	1.0326	1.0003	1.0033	1.0326
	0.0000	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0000*
0.01	0.0114	0.1128	0.7546	0.0124	0.1228	0.7887
	1.5294	*1.4280*	*0.7197*	*1.5394*	*1.4291*	*0.6815*
0.02	0.0030	0.0299	0.2864	0.0032	0.0316	0.3040
	1.5294	*1.5024*	*1.2487*	*1.5555*	*1.5272*	*1.2593*
0.03	0.0014	0.0136	0.1352	0.0014	0.0141	0.1411
	1.5468	*1.5344*	*1.4151*	*1.5620*	*1.5494*	*1.4265*
0.04	0.0008	0.0077	0.0772	0.0008	0.0079	0.0801
	1.5578	*1.5509*	*1.4832*	*1.5653*	*1.5583*	*1.4894*
0.05	0.0005	0.0049	0.0496	0.0005	0.0051	0.0514
	1.5654	*1.5610*	*1.5183*	*1.5672*	*1.5628*	*1.5198*
0.06	0.0003	0.0034	0.0344	0.0004	0.0035	0.0357
	1.5706	*1.5677*	*1.5393*	*1.5682*	*1.5653*	*1.5369*
0.07	0.0002	0.0025	0.0252	0.0003	0.0026	0.0262
	1.5739	*1.5720*	*1.5526*	*1.5686*	*1.5667*	*1.5474*
0.08	0.0002	0.0019	0.0192	0.0002	0.0020	0.0200
	1.5757	*1.5744*	*1.5613*	*1.5685*	*1.5672*	*1.5542*
0.09	0.0002	0.0015	0.0151	0.0002	0.0016	0.0158
	1.5762	*1.5753*	*1.5667*	*1.5679*	*1.5670*	*1.5585*
0.10	0.0001	0.0012	0.0122	0.0001	0.0013	0.0128
	1.5756	*1.5750*	*1.5697*	*1.5670*	*1.5665*	*1.5612*
0.20	0.0000	0.0003	0.0030	0.0000	0.0003	0.0032
	—	*1.5573*	*1.5580*	—	*1.5525*	*1.5533*
0.40	0.0000	0.0001	0.0007	0.0000	0.0001	0.0008
	—	*1.5126*	*1.5126*	—	*1.5098*	*1.5098*
0.60	0.0000	0.0000	0.0003	0.0000	0.0000	0.0004
	—	—	*1.4589*	—	—	*1.4570*

Table 4. Values of the modulus and of the argument (in italic) of $Bi\psi$ at $s = 0$ for a tungsten conductor with $\Delta = 0.4$ and the region $0 \leqslant s \leqslant 0.4$ occupied either by a Pyrex glass or by Teflon

	$\Gamma = 0.00890$; $\Lambda_d/\Omega = 396.2$ (Pyrex glass)			$\Gamma = 0.00159$; $\Lambda_d/\Omega = 584.4$ (Teflon)		
Ω	$Bi = 10^{-3}$	$Bi = 10^{-2}$	$Bi = 10^{-1}$	$Bi = 10^{-3}$	$Bi = 10^{-2}$	$Bi = 10^{-1}$
0.00	1.0003	1.0033	1.0326	1.0003	1.0033	1.0326
	0.0000	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0000*
0.01	0.0104	0.1030	0.6892	0.0088	0.0875	0.5620
	2.1325	*2.0311*	*1.3228*	*2.7157*	*2.6053*	*1.8577*
0.02	0.0014	0.0140	0.1339	0.0006	0.0062	0.0593
	3.3401	*3.3131*	*3.0594*	*4.4478*	*4.4195*	*4.1516*
0.03	0.0003	0.0025	0.0252	0.0001	0.0006	0.0065
	4.4953	*4.4830*	*4.3637*	*6.1141*	*6.1015*	*5.9785*
0.04	0.0001	0.0005	0.0054	0.0000	0.0001	0.0008
	5.6318	*5.6249*	*5.5572*	—	*1.4839*	*1.4150*
0.05	0.0000	0.0001	0.0013	0.0000	0.0000	0.0001
	—	*0.4760*	*0.4333*	—	—	*3.1005*
0.06	0.0000	0.0000	0.0003	0.0000	0.0000	0.0000
	—	—	*1.5771*	—	—	—

averaged dimensionless temperature, while the phase tends to zero.

The temperature field can be easily evaluated from the dimensionless temperature values by employing equation (41) provided that Q is prescribed. If Q is not prescribed but the effective electric current I_{eff} is known, then equation (41) can still be employed because Q is determined by I_{eff} through the relation

$$Q = R_0 \varphi(\Omega, \Delta) I_{eff}^2 \tag{93}$$

Table 5. Values of the modulus and of the argument (in italic) of $Bi\psi$ at $s = 1$ for a tungsten conductor with adiabatic conditions at $s = \Delta$

	$\Delta = 0.4$			$\Delta = 0.7$		
Ω	$Bi = 10^{-3}$	$Bi = 10^{-2}$	$Bi = 10^{-1}$	$Bi = 10^{-3}$	$Bi = 10^{-2}$	$Bi = 10^{-1}$
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	0.0000	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0000*
0.01	0.0129	0.1273	0.7838	0.0212	0.2070	0.9026
	1.5578	*1.4430*	*0.6699*	*1.5494*	*1.3620*	*0.4447*
0.02	0.0032	0.0321	0.3009	0.0053	0.0528	0.4643
	1.5672	*1.5383*	*1.2645*	*1.5645*	*1.5170*	*1.0869*
0.03	0.0014	0.0143	0.1389	0.0024	0.0235	0.2269
	1.5685	*1.5556*	*1.4300*	*1.5664*	*1.5453*	*1.3398*
0.04	0.0008	0.0080	0.0786	0.0013	0.0132	0.1300
	1.5686	*1.5613*	*1.4895*	*1.5661*	*1.5542*	*1.4370*
0.05	0.0005	0.0051	0.0504	0.0008	0.0085	0.0836
	1.5682	*1.5634*	*1.5164*	*1.5650*	*1.5574*	*1.4819*
0.06	0.0004	0.0036	0.0351	0.0006	0.0059	0.0582
	1.5675	*1.5640*	*1.5302*	*1.5634*	*1.5581*	*1.5054*
0.07	0.0003	0.0026	0.0258	0.0004	0.0043	0.0428
	1.5666	*1.5639*	*1.5377*	*1.5615*	*1.5576*	*1.5186*
0.08	0.0002	0.0020	0.0197	0.0003	0.0033	0.0328
	1.5656	*1.5634*	*1.5419*	*1.5593*	*1.5563*	*1.5262*
0.09	0.0002	0.0016	0.0156	0.0003	0.0026	0.0259
	1.5644	*1.5625*	*1.5440*	*1.5569*	*1.5545*	*1.5304*
0.10	0.0001	0.0013	0.0127	0.0002	0.0021	0.0210
	1.5631	*1.5614*	*1.5450*	*1.5543*	*1.5523*	*1.5325*
0.20	0.0000	0.0003	0.0032	0.0001	0.0005	0.0052
	—	*1.5417*	*1.5337*	*1.5190*	*1.5182*	*1.5107*
0.40	0.0000	0.0001	0.0008	0.0000	0.0001	0.0013
	—	*1.4701*	*1.4663*	—	*1.4182*	*1.4143*
0.60	0.0000	0.0000	0.0002	0.0000	0.0000	0.0003
	—	—	*1.3124*	—	—	*1.2134*

where $R_0 = 1/[\pi\sigma(b^2-a^2)]$ is the resistance of the electric conductor per unit length and for a stationary current, while function φ is defined as

$$\varphi(\Omega,\Delta) \equiv \frac{(1-\Delta^2)\displaystyle\int_\Delta^1 \|f(s',\Omega,\Delta)\|^2 s'\,\mathrm{d}s'}{2\left\|\displaystyle\int_\Delta^1 f(s',\Omega,\Delta)s'\,\mathrm{d}s'\right\|^2}. \tag{94}$$

Equations (93) and (94) can be obtained by a method similar to that employed in [13] for a resistor in the form of a solid cylinder.

CONCLUSIONS

The power generated per unit volume by the Joule effect in an infinitely long and hollow cylindrical conductor crossed by an alternating current has been evaluated by taking into account the skin effect and assuming that the internal hole is filled with a dielectric solid. Then, the Fourier equation has been written in a dimensionless form both in the region occupied by the electric conductor and in that occupied by the dielectric. The dimensionless temperature field has been determined analytically in a steady periodic regime. Values of the time-average, amplitude and phase of the dimensionless temperature field have been reported in tables. The amplitude and the phase have been evaluated with reference to two kinds of dielectric: Pyrex glass and Teflon. Three distributions of the time-averaged and dimensionless temperature within the electric conductor have been plotted, with increasing values of the frequency. It has been pointed out that the time-averaged and dimensionless temperature tends to become uniformly distributed both in the conductor and in the dielectric when the electric current frequency tends to infinity. The amplitude and the phase of the dimensionless temperature oscillations have been evaluated when the ratio Δ between the internal and external radii of the hollow cylinder is 0.4. It has been shown that the amplitude of the dimensionless temperature oscillations decreases very rapidly to zero as the frequency increases, so that for a tungsten conductor with an external radius of 1 cm the amplitude is negligible for a frequency greater than 33.3 Hz. Finally, the case in which the hole within the conductor is occupied by empty space has been considered. The numerical values of the amplitude and phase of the dimensionless temperature oscillations have been evaluated for $\Delta = 0.4$ and $\Delta = 0.7$. The results for $\Delta = 0.4$ are similar to those obtained

Table 6. Values of the modulus and of the argument (in italic) of $Bi\psi$ at $s = \Delta$ for a tungsten conductor with adiabatic conditions at $s = \Delta$

	$\Delta = 0.4$			$\Delta = 0.7$		
Ω	$Bi = 10^{-3}$	$Bi = 10^{-2}$	$Bi = 10^{-1}$	$Bi = 10^{-3}$	$Bi = 10^{-2}$	$Bi = 10^{-1}$
0.00	1.0003	1.0033	1.0326	1.0002	1.0016	1.0157
	0.0000	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0000*
0.01	0.0129	0.1277	0.8093	0.0212	0.2073	0.9168
	1.5578	*1.4431*	*0.6701*	*1.5494*	*1.3620*	*0.4447*
0.02	0.0032	0.0322	0.3107	0.0053	0.0529	0.4716
	1.5672	*1.5384*	*1.2652*	*1.5645*	*1.5170*	*1.0870*
0.03	0.0014	0.0143	0.1434	0.0024	0.0235	0.2305
	1.5685	*1.5558*	*1.4315*	*1.5664*	*1.5453*	*1.3400*
0.04	0.0008	0.0080	0.0812	0.0013	0.0132	0.1320
	1.5686	*1.5615*	*1.4922*	*1.5661*	*1.5543*	*1.4373*
0.05	0.0005	0.0052	0.0520	0.0008	0.0085	0.0849
	1.5682	*1.5638*	*1.5206*	*1.5650*	*1.5574*	*1.4825*
0.06	0.0004	0.0036	0.0362	0.0006	0.0059	0.0591
	1.5676	*1.5647*	*1.5362*	*1.5634*	*1.5582*	*1.5062*
0.07	0.0003	0.0026	0.0266	0.0004	0.0043	0.0434
	1.5668	*1.5648*	*1.5456*	*1.5615*	*1.5577*	*1.5197*
0.08	0.0002	0.0020	0.0203	0.0003	0.0033	0.0333
	1.5659	*1.5646*	*1.5516*	*1.5593*	*1.5564*	*1.5276*
0.09	0.0002	0.0016	0.0160	0.0003	0.0026	0.0263
	1.5649	*1.5641*	*1.5556*	*1.5569*	*1.5546*	*1.5322*
0.10	0.0001	0.0013	0.0130	0.0002	0.0021	0.0213
	1.5639	*1.5633*	*1.5581*	*1.5543*	*1.5525*	*1.5346*
0.20	0.0000	0.0003	0.0032	0.0001	0.0005	0.0053
	—	*1.5507*	*1.5515*	*1.5199*	*1.5197*	*1.5176*
0.40	0.0000	0.0001	0.0008	0.0000	0.0001	0.0013
	—	*1.5088*	*1.5088*	—	*1.4281*	*1.4284*
0.60	0.0000	0.0000	0.0004	0.0000	0.0001	0.0006
	—	—	*1.4562*	—	*1.3333*	*1.3333*

in the case in which the hole within the conductor is filled with Teflon.

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